ISBN: 978-981-18-5852-9
WCSE 2022 Spring Event: 2022 9th International Conference on Industrial Engineering and Applications
doi: 10.18178/wcse.2022.04.048

Mathematical Modeling for the Break-Even Point Problem in a Non-homogeneous System

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Abstract. This paper presents a mathematical formulation for the break-even problem in a non-homogeneous system. The optimization problem aims to obtain the composition of the best product mix in a non-homogeneous industrial plant, with the lowest cost until the Break-Even Point is reached. The constraints of the problem represent real constraints of a generic inhomogeneous industrial plant for n distinct products. The proposed model can solve the break-even problem simultaneously for all products, unlike existing approaches that propose a sequential solution, considering each product in isolation and providing a suboptimal solution to the problem. The results indicate that the product mix found through the proposed model has economic advantages over the traditional approach used.

Keywords:branch and bound, break-even point, non-homogeneous production system, integer linear programming, management accounting.

1. Introduction

The Break-Even Point (BEP) indicates the level of production at which profit is null [1] [2] [3] [4] [5] and, therefore, is associated with a product mix in which the total cost is equivalent to the total revenue from the sale of products. The mathematical analysis of the Break-Even Point is useful to assist managers in making short and medium-term decisions since BEP provides the manager with the minimum quantities of each product necessary to balance the fixed costs of production and the unit costs of each of the products. their respective recipes [1][2][3]. In general, the BEP is obtained directly from the Sales Price (PV), the Variable Unit Cost (CV_u), and the Fixed Cost (CF). Average values of these quantities are obtained for each product in isolation, which is not appropriate for the analysis of a production line with the capacity to process more than one type of product.

In this work a generic model of Integer Linear Programming (PLI) is proposed for the Break-Even Point problem that can be applied to several products simultaneously. The main objective is to obtain an optimum break-even value in a non-homogeneous production system so that the total production cost up to the BEP is less than that obtained by traditional approaches [1][2][4][5]. The proposed approach allows the determination of the BEP in a non-homogeneous system associated with a total variable cost lower than that obtained in the traditional approach, this directly implies in increasing the competitiveness of the production system, since after the BEP, any linear combination of the product mix will lead to profit, since fixed costs have already been paid and the selling price will be higher than the variable unit cost.

Some proposals for optimization models in non-homogeneous systems are presented in the literature. González (2001) [6], the author develops an alternative model for multi-product cost-volume-profit based on ABC systems, in which the Break-Even Point is obtained based on the main products (best-selling or with the highest profit margin). This approach is justified to the extent that the greater the contribution margin of

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the product, the more quickly the fixed cost of the plant will be amortized. Stefan and Stefan (2008)[4], propose a model based on profit optimization. An approach based on the product mix is presented by [7] in which the contribution margins of each product are optimized, which also leads to profit maximization. Tsai and Lin (1990) [8] present a binary mixed integer programming model for the non-linear cost-volume-profit analysis of various products using a selective approach. Kucharski and Wywial [3] propose two models. The first comprises the maximization of unit profit and aims to anticipate the amortization of the fixed cost, thus reducing the number of units necessary to achieve the BEP. In this case, a sub-optimal approach is adopted through multicriteria analysis and the Manhattan similarity metric, aiming to bring the BEP close to zero. The second model is based on minimizing the variable cost of production resulting in a system of differential equations that allows a solution for a specific situation in which the ratio between the contribution margin and the cost is high.

The main contribution of this work consists of proposing a model in which all available products are used to amortize the fixed cost, thus resulting in an optimal combination of the product mix. The results obtained show that the proposed optimization model presents significant economic advantages in relation to the current model, generating an average saving of 16.34% for the analyzed system, in addition to the reduction in the number of units necessary to reach the BEP. This work shows, for the first time, the determination of the equilibrium point represented by units of different types of products and the proposed model can be applied in other non-homogeneous systems.

2. Methodology

2.1. Inhomogeneous Production

The total cost of production in a homogeneous system is given by:

$$CT = CF_t + (q * CV_u) \tag{1}$$

where CF_t is the total fixed cost, CV_u is the variable unit cost and q is the number of units produced.

If the total revenue (derived exclusively from sales) is equal to the total cost of production, the profit is zero and the production equilibrium is reached:

$$q_{bp} = \frac{CF_t}{(PV - CV_u)} \tag{2}$$

where PV is the selling price and q_{bp} it is the balance point for a homogeneous system.

The determination of the Break-Even Point in a non-homogeneous production line can be analyzed based on an optimization problem. In this case, the total cost of production and the total revenue are given by:

$$CT = CF_t + \sum_{i=1}^{n}(q_i * CV_{ui}) \tag{3}$$

$$RT = \sum_{i=1}^{n}(q_i * PV_i) \tag{4}$$

whereCV_{u_i}, PV_i e q_i are the unit variable cost, the selling price and the quantity produced of the product i $(i = 1, \dots, n)$, respectively.

From (3) and (4) the following relations are reached between the equilibrium point associated with each of the products:

$$\sum_{i=1}^{n}\left[q_{bp,i} * (PV_i - CV_{ui})\right] - CF_t \tag{5}$$

The equation (5) it provides multiple solutions for the balance points of the products to be sold, which suggests the search for the best solution, obeying the restrictions inherent to the production and commercialization process. These restrictions are associated with the production capacity and demand for products by the consumer market and must be considered as strict limits of minimum (market demand) and maximum (production capacity) for each of the equilibrium points$q_{bp,i}$, i $(i = 1, \dots, n)$.

Another important aspect that must be taken into account when modelling this problem is the restrictions on production capacity and demand for products, as it would make no sense to obtain a result that is above the factory's production capacity, this would make it impossible to obtain the great results in practice. Likewise, it is necessary to meet market demands, so production must have a minimum required value, limiting the problem to a defined interval, with a minimum value equal to market demand and maximum value equal to production capacity. In this way, as the individual quantity of products to be manufactured in order to reach the Break-Even Point cannot exceed the production capacity, as follows:

$$q_{bp,i} \leq k_i \tag{6}$$

Where k_i represents the maximum production capacity column vector for each of its products. Analogously to what was done with the values of individual capacities, it is possible to represent the constraints of individual demands by the vector, as follows:

$$\lambda \cdot d_i \leq q_{bp,i} \tag{7}$$

The values of vector d_i can be obtained through demand forecasting techniques using historical data of the organization's sales with a significantly representative time interval and with relative confidence. A relevant fact in relation to the demand constraint is that this cannot be the real demand, since, if the Break-Even Pointis above the real average demand, it cannot be reached, there is no longer a demand for it. A solution to this problem is to use a correction factor λ, such that $\lambda \in (0,1)$ multiplying the vector d, this artifice will make the vector λd_i move "below" the real average demand, as a consequence the Break-Even Pointwill be located within the region where the real average demand is.

The market demand associated with each product has an intrinsically dynamic and periodic character. The approach proposed in this work consists of considering the demand for each product as an average of the sales forecast over a period of time that represents a seasonal pattern of behaviour. In this case, it is important to ensure that the lower demand limit for each product is not greater than the actual demand practiced at any given time, which would imply in achieving an unrealizable break-even point.

With these elements in mind, it is then possible to establish a constraint that contemplates the upper and lower limits of the problem in question, as follows:

$$\lambda \cdot d_i \leq q_{bp,i} \leq k_i (i = 1, \dots, n) \tag{8}$$

The objective function is to minimize the total variable cost, having the following optimization problem with entire decision variables:

$$Min\, J(q_{bp,1}, \dots, q_{bp,2}) \sum_{i=1}^{n} \left(q_{bp,i} \cdot CV_{u_i}\right) \tag{9}$$

$$\sum_{i=1}^{n} [q_{bp,i} \cdot (PV_i - CV_{u_i})] - CF_t = 0 \tag{10}$$

$$\lambda \cdot d_i \leq q_{bp,i} \leq k_i (i = 1, \dots, n)$$

$$q_{bp,i}, k_i, d_i \in Z \qquad (i = 1, \dots, n)$$

$$\lambda \in (0,1)$$

k_i and d_i are the maximum capacities and average market demand associated with the product $(i = 1, \dots, n)$ and λ it is a parameter to be specified in such a way that the real demand for each product is met over a given period.

The value of λ plays a very important role in the model, as changing the value of this parameter changes the configuration of the BEP and, consequently, the value of the objective function at the optimal point (minimization of Total Cost), the value of the contraction factor λ adopted was 0.25 for all products. From the tests carried out, it was possible to notice that when approaching λ to 1, the problem has no solution, which is consistent with the hypothesis that there cannot be a BEP greater than the demand. Another way of interpreting this result is by analyzing the region of feasible solutions, as λ approaches 1 the number of viable solutions tends to zero. The highest value of λ admitted by the model was 0.9 and a value above this limit results in a problem with no viable solution. Similarly, by approaching the parameter λ to zero, the solution of the problem tends to increase the amount of product with a higher contribution margin and minimize the amount of the others, which is predicted by the equality constraint of the BEP that aims to dampen the fixed cost. In this way, the product with the highest contribution margin and highest production starts to have a greater impact on fixed cost amortization.

3. Numerical Results and Conclusion

The case study was done in a small company in the furniture segment. The company in question is located in the euro zone, has a plant with 3 production lines (Alder, Cherry and Walnut) and a total of 9 products, (Alexa, Alica, Aurélia, Elsa, Ema, Erika, Daniela, Diana and Denisa), with a maximum production capacity of 5030 units per month.

From the data extracted from [4] and adapted in Table 1, it was then possible to calculate the Break-Even Pointthrough the variable cost method and also through the proposed modelling.

Table 1: Data from production line

Production Lines	Products	Sale price	Unit cost
Alder	Alexa	€ 35.00	€ 15.00
	Alica	€ 40.00	€ 20.00
	Aurélia	€ 50.00	€ 30.00
Cherry	Elsa	€ 25.00	€ 17.00
	Ema	€ 25.00	€ 17.00
	Erika	€ 25.00	€ 17.00
Walnut	Daniela	€ 36.00	€ 32.40
	Diana	€ 32.00	€ 28.80
	Denisa	€ 34.00	€ 30.60

Source: Adapted from [2].

The results are presented in Tables 4 and 5 and show the BEP by product and by production line. The scenarios presented relate to changes in the maximum quantities produced of a given product on the production line, considering that the maximum capacity of the factory is 5030 units. Increasing the production of one type of product will imply a reduction in the production of another due to the physical limitations of the process. This fact changes the values of the break-even calculation, thus changing the final results for each product individually, the configuration for the possible scenarios, as well as for the initial condition of the manufacturing layout are shown in Table 2.

Table 2: Possible Production Settings

Products	Initial Condition	Scenario 1	Scenario 2	Scenario 3
Alexa	1100.00	200.00	200.00	1100.00
Alica	250.00	400.00	400.00	250.00
Aurélia	50.00	800.00	800.00	50.00
Elsa	400.00	1000.00	400.00	1000.00
Ema	450.00	100.00	450.00	100.00
Erika	380.00	130.00	380.00	130.00
Daniela	500.00	500.00	1200.00	1200.00
Diana	1500.00	1500.00	300.00	300.00
Denisa	400.00	400.00	900.00	900.00
$\sum$	5,030.00	5,030.00	5,030.00	5,030.00

Source: Adapted from [2].

The data used in the calculation of the BEP by the proposed modeling for the initial condition and the other scenarios are shown in Table 3, and relate to the vector λd_i and the parameter k_i for each of the product.

Table 3: Demand factor and maximum capacity

Products	λd				
	Initial Condition	*Scenario 1*	*Scenario 2*	*Scenario 3*	*Maximum capacity*
Alexa	50.00	275.00	50.00	50.00	1100.00
Alica	100.00	62.50	100.00	100.00	400.00
Aurélia	200.00	12.50	200.00	200.00	800.00
Elsa	100.00	100.00	250.00	100.00	1000.00
Ema	112.50	112.50	25.00	112.50	450.00
Erika	95.00	95.00	32.50	95.00	380.00
Daniela	125.00	125.00	125.00	300.00	1200.00
Diana	375.00	375.00	375.00	75.00	1500.00
Denisa	100.00	100.00	100.00	225.00	900.00

The calculation of the Equilibrium Point by the proposed model used a lambda equal to 0.25 for both the calculation of the initial condition and for the other scenarios. This choice was made after simulations for several values of this parameter, the value of 0.25 being the result with a more uniform distribution of the products. Table 4 summarizes the results obtained by the standard model used, both for the initial condition, as well as for the other possible scenarios for production.

Table 4: Break-Even Point(Standart method)

Products	Initial Condition		Scenario 1		Scenario 2		Scenario 3	
BEP ($)[a]	*€ 73.60*		*€ 53.60*		*€ 65.60*		*€ 21.80*	
BEP (und)[b]	*1654*		*1653*		*1653*		*1637*	
	und	∑	und	∑	und	∑	und	∑
Alexa	66		361		66		65	
Alica	132	461	82	460	132	461	130	455
Aurélia	263		17		263		260	
Elsa	132		132		328		130	
Ema	148	405	148	405	33	404	147	401
Erika	125		125		43		124	
Daniela	164		164		164		390	
Diana	492	788	492	788	492	788	98	455
Denisa	132		132		132		293	

Source: Adaptedfrom[2]

[a]BEP ($) – Residual value after reaching break-even point.

[b]BEP (und) – Total units of products.

Σ – Sum of BEP for each production line made up of 3 products.

Table 4 shows the Break-Even Pointcalculated from the standard method which considers each product separately.

Table 5 shows the Break-Even Pointcalculated from the model developed in the research and proposed in this article. The results in Table 5 show a significant reduction in the “BEP (und)” values, about 200 units, as well as meeting the theoretical constraint of equation (7) presented in the table by the BEP ($) line. The Break-Even Pointis served both in the initial production condition and in the other scenarios. Table 6 shows a comparison between the total variable costs of production up to the Break-Even Pointobtained in each of the approaches.

Table 5: Break-Even Point (proposed model with λ = 0.25)

Products	Initial Condition		Scenario 1		Scenario 2		Scenario 3	
BEP ($)[c]	*€ 00.00*		*€ 00.00*		*€ 00.00*		*€ 00.00*	
BEP (und)[d]	*1454*		*1451*		*1454*		*1434*	
	und	∑	und	∑	und	∑	und	∑
Alexa	216		433		216		159	
Alica	100	516	72	518	100	516	162	521
Aurélia	200		13		200		200	
Elsa	100		100		250		100	
Ema	113	335	113	330	25	335	113	311
Erika	122		117		60		98	
Daniela	126		126		126		301	
Diana	216	603	433	603	216	603	159	602
Denisa	100		72		100		162	

[c]BEP ($) – Residual value after reaching break-even point.

[d]BEP (und) – Total units of products.

Σ – Sum of BEP for each production line made up of 3 products

Table 6: Total variable costs of production

	Variable Costing	Proposed Modeling	Absolute Difference	Relative Difference
Initial	€ 41,927.40	€ 34,935.00	€ 6,992.40	16.68%

Condition				
Scenario 1	€ 37,972.40	€ 31,935.00	€ 6,037.40	15.90%
Scenario 2	€ 41,910.40	€ 34,935.00	€ 6,975.40	16.64%
Scenario 3	€ 42,616.20	€ 35,740.00	€ 6,876.20	16.14%

It is possible to see a reduction in the total variable production costs up to the BEP for the initial condition and other scenarios. The proposed model offers an average result 16.34% more economical than the standard approach [4]. This gain can be converted into gross profit since, after the break-even point, any product mix will generate a profit since the fixed costs have already been reduced and the sale price of each product is always higher than the unit cost of its production.

4. Conclusions

In addition to offering economic advantages over the standard approach, the proposed model is able to provide the production manager with an insight into the competitiveness behavior between products on the production line, being an important support for more efficient operational planning, reduction setup times, proper planning of acquisition and use ofraw materials and reduction of unnecessary movements. Prior knowledge of the Break-Even Pointalso provides important information for the sales team, making it possible to forecast what to sell and how much to sell for each product, in addition to efficient planning of marketing strategies associated with the planning of sales goals.

Another relevant aspect of this modeling is the lambda correction factor, this correction factor appears in the modeling as an artifice that enables the solution of the problem, in other words, without this correction factor the problem would not have a viable solution and would fall into the same problem of previous modeling [8] [9] [10] [11] where it was possible to express this problem as an optimization problem, but that contained infinite solutions, which for industrial and business use does not have many applications and advantages, in general, entrepreneurs look for unique and economically viable. the addition of this correction factor is also a relevant scientific contribution, as it raises questions about the behavior of the BEP as a function of its value, as stated throughout the text, lambda values closer to zero tend to funnel the solution into a single product (which has the highest contribution margin) and neglecting the production of other products, on the other hand, solutions closer to 1 tend to "uniform" the distribution of products better, so future works can explore the optimal value of lambda in relation to some other objective function, thus obtaining a more optimized distribution, but not necessarily at a lower cost.

Finally, another suggestion for future work is to use stochastic variables for the values of sales price and unit variable cost and to evaluate the behavior of the BEP of each product for different values of these variables, thus obtaining a BEP as an interval instead of a single and fixed value as presented in this work, this would make the solution closer to the realization of companies, as the costs involved in the manufacture of products undergo constant changes due to the monetary exchange that directly impacts the costs of acquiring raw material from other countries, as well as changes in the sales price that occur as a result of negotiations between the interested parties.

5. References

[1] D. M. Dutta. *Cost Accounting: Principles And Practice*. Pearson Education India, Delhi, 2004.

[2] R. Morabito, V. Pureza, A. Fleury, C. H. P. Mello, D. N. Nakano, de Lima, E. P., ... and R. Sousa. *Metodologia de pesquisa em engenharia de produção e gestão de operações*. Elsevier Brasil, 2018.

[3] R. Kucharski and J. L. Wywial. Optimization of the break-even point for non-homogeneous products sales. *Stowarzyszenie Księgowych w Polsce*. 2019, **101**(157): 133-148.

[4] D. Stefan, A. B. Stefan, L. Savu, R. Sumandea, C. A. Comes. A Cost-Volume-Profit Model for A Multiproduct Situation with Variable Production Structure. *20th EURO Mini Conference Continuous Optimization and Knowledge-Based Technologies Neringa, Lithuania*. 2008, pp. 349–352.

[5] R. Ravichandran. A Decision Support System for Stochastic Cost-Volume-Profit Analysis.*Decis. Support Syst.* 1993, **10** (4): 379-399.

[6] L. González. Multiproduct CVP analysis based on contribution rules.*Int. J. Prod. Econ*. 2001, **73** (13): 273-284.

[7] M. Potkany and L. Krajcirova. Quantification of the Volume of Products Achieve the Break-Even Point and Desired Profit in Non-Homogeneous Production. *Procedia Econ. Financ*. 2015, **26**: 194-201.

[8] W. H. Tsai and T. M. Lin.Nonlinear Multiproduct CVP Analysis with 0-1 Mixed Integer Programming. *Eng. Cost. Prod. Econ*. 1990, **20** (1): 81-91.

[9] Z. Pawłowski.*Ekonometryczna analiza procesu produkcyjnego*. PWN: Warszawa, 1970.

[10] R.S. Kaplan and A.A. Atkinson. *Advanced Management Accounting*. 3nd ed. Prentice Hall, Upper Saddle River, NJ, 1998.

[11] B. Pomykalska.*Wykorzystanie dźwigni w planowaniu zysku i ocenie ryzyka*. PWN: Warszawa, 2007.